MENTAL OPERATIONS FOR ALTERING LENGTH AND PRESERVING ANGULARITY

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Angularity is a persistent quantity throughout K–12+ school mathematics, and many studies have shown that individuals often conflate angularity with linear attributes (e.g., the length of an angle model's sides). However, few studies have examined the productive ways in which students might reason about angularity while attending to linear attributes like side lengths. Leveraging data from a yearlong teaching experiment with ninth-grade students, I present four mental operations that students indicated for altering lengths while preserving angularity. Additionally, I consider implications of these mental operations for teaching and research.

Keywords: Geometry and Spatial Reasoning, Cognition

Angularity is a persistent quantity in K–12+ school mathematics (Barabash, 2017), and considerably less has been written about individuals' thinking regarding angular measure compared to other quantities (Smith & Barrett, 2017). Furthermore, scholars have explicitly called for additional research into how students quantify angularity (e.g., Moore, 2013; Smith & Barrett, 2017). Perhaps in response to these calls or the curricular salience of angularity, in recent years several researchers have conducted studies foregrounding angularity (e.g., Alyami, 2022; Germia, 2022; Hardison, 2018; Mullins, 2020). Despite this recent increase in studies and researchers focused on angularity, there remains a need for basic research elaborating the mental operations that play a role in individuals' quantifications of angularity. For example, several studies discussed in the subsequent section have shown that individuals often problematically conflate angularity with linear attributes; however, little has been reported regarding whether and how individuals might productively attend to the length of an angle model's sides, for example, when reasoning about angularity. This report offers progress toward this end by examining some ways in which students can productively attend to both linear and angular attributes. In particular, this study addresses the following question: What mental operations do students enact to preserve angularity while altering length?

Some Relevant Extant Literature

Students' tendencies to conflate angularity with numerous attributes (e.g., orientation, area, length, etc.) have been documented (Smith & Barrett, 2017). Here, I present a brief summary of research reporting on conflations involving length and angularity. In studies with students in elementary and middle grades, researchers have found students often conflate angularity with other attributes including two primary variations of linear attributes. First, students often judge which of two angles is larger by comparing the lengths of the angle model's sides (Baya'a et al., 2017; Clements et al., 1996; Crompton, 2017; Devichi & Munier, 2013; Keiser, 2004; Lehrer et al., 1998). Second, students also tend to consider angularity to refer to the linear distance between an angle model's sides (Baya'a et al., 2017; Keiser, 2000; 2004; Lehrer et al., 1998; Thompson, 2013); I refer to this second linear attribute as the *span* of an angle model (see Figure 0 wherein solid and dotted lines indicate perceptually available and absent segments, respectively). For example, when 60 elementary children enrolled in a three-year longitudinal study were asked to measure the amount of opening in various contexts including hinged wooden

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jaws, an open door, or a bent straw, the children measured the span of these angle models 95% of the time (Lehrer et al., 1998). Lehrer and colleagues posited that children believed measuring length was “an adequate measure of angle” (p. 151), and they also found that conflating length and angularity was a persistent issue throughout the longitudinal study remarking, “the effects of length on children’s judgments about angles did not diminish during the 3 years of the study” (p. 149).

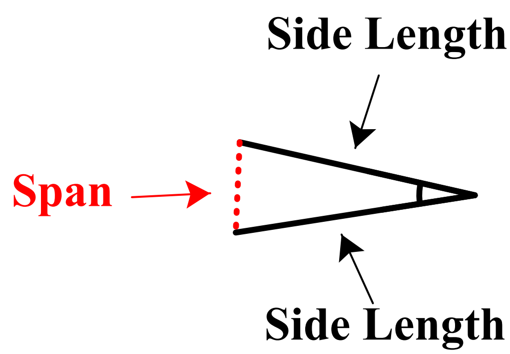

Figure 0: Two Linear Attributes of an Angle Model: Side Length and Span

Brief Theoretical Considerations

The study at hand was heavily informed by a radical constructivist epistemology (von Glasersfeld, 1995) and principles of quantitative reasoning (Thompson, 1994; 2011). According to Thompson, an individual has constructed a quantity when they have conceived an attribute of an object along with an imagined process for measuring the attribute; such an imagined measurement process necessarily involves a mental comparison of at least two instantiations of the attribute at hand. Angularity is one example of an attribute that an individual might quantify; other such attributes include length, area, time, and speed. Because quantities exist in individuals’ minds, investigating how individuals conceive quantities necessarily involves attending to the mental operations particular attributes permit (from the perspective of that individual); mental operations include, for example, imagined physical actions like partitioning and iteration (Steffe & Olive, 2010). The collection of mental operations an attribute permits for an individual constitutes the individuals’ quantification for that attribute. Thus, quantities and quantifications necessarily vary, both across individuals and within an individual over time.

Two Opposing Hypotheses Regarding Length and Angularity

Given individuals’ propensities for conflating length and angularity, a reasonable hypothesis (at least at first blush) is that quantifications of length serve as obstacles for productively quantifying angularity. I refer to this hypothesis as *the angular interference hypothesis* (Hardison, in press). Educators working under the angular interference hypothesis might, for example, approach early angular-measure instruction in such a way as to divorce (as best as possible) students’ ways of thinking length and angularity. A second hypothesis, *the angular reorganization hypothesis*, stands in opposition to the angular interference hypothesis. The angular reorganization hypothesis posits that students’ quantifications of length can be used in service of quantifying angularity. Educators working under the angular reorganization hypothesis might, therefore, establish instructional goals different from those who prescribe to the interference hypothesis.

For the present study, the angular reorganization hypothesis was adopted. One reason for this choice involves considering students early experiences with angular measure; (see Hardison, 2018 for other reasons). Early experiences with angles often involve perceptually available angle models with finite side lengths (e.g., two drawn line segments sharing a common vertex or a pair of hinged wood chopsticks). As intimated in a previous section, conceiving of angular measure involves at least an implicit consideration of two instantiations of angularity, and two perceptually available angle models need not have sides of the same length. Because comparing extents of angularity across two such perceptually available angle models would require a cognitive means of rendering differences in length inconsequential, I conjectured that angularity-preserving, length-altering mental operations might play a critical role in individuals' initial quantifications of angularity. This conjecture is consistent with the angular reorganization hypothesis, and operations of this nature are the focus of this report.

Methods

The data presented in the following sections was curated from a teaching experiment (Steffe & Thompson, 2000; Steffe & Ulrich, 2013) conducted in the southeastern U.S. with two ninth-grade students, Camille and Kacie, and transpiring over a single scholastic year. During the study, both students (a) were enrolled in a first-year algebra course and (b) had yet to experience a dedicated geometry course. The primary aim of the teaching experiment was to investigate how the students quantified angularity as well as the progressions transpiring in these quantifications during the study (see Hardison, 2018). The author served as teacher-researcher throughout the teaching experiment. During the study, students discussed their thinking as they interpreted and addressed mathematical tasks involving angle models, which included physical manipulatives (e.g., hinged wooden chopsticks), drawings, and virtual representations in dynamic geometry environments.

Each student participated in 2 initial interview sessions and 1 final interview session. Between these initial and final sessions, Camille and Kacie participated in 11 and 10 teaching sessions, respectively, which were conducted approximately once per week outside of their regular classroom instruction. Each session was video-recorded and approximately 30 minutes in length. Interview sessions were conducted with each student individually to understand (and not intentionally occasion change in) their ways of reasoning at the beginning and end of the teaching experiment. In contrast, during teaching sessions the teacher-researcher worked to both understand students' existing ways of reasoning and to engender productive changes in students' ways of reasoning; teaching sessions were conducted individually or in pairs. Data sources included recordings, digitized student work, and field notes. Conceptual analysis (Thompson, 2008; von Glasersfeld, 1995) was used to scrutinize records of students' observable activities (e.g., talk, gestures, written responses, etc.) both during the teaching experiment (on-going analysis) and afterwards (retrospective analysis). This report focuses on length-altering, angularity-preserving mental operations.

Findings

From analyzing students' observable activities in the teaching experiment, I abstracted four mental operations the students enacted to alter linear attributes while preserving angularity. In the subsequent sections, I illustrate each of these mental operations using selected data from Kacie's first three teaching sessions, the first two of which were paired teaching sessions with Camille. These three teaching sessions were selected for this report because they were the first sessions in which students indicated length-altering, angularity-preserving mental operations.

The First Mental Operation: Truncation (Teaching Session #1)

During Kacie and Camille's first paired teaching session, two pairs of hinged wooden chopsticks had been set to different extents of angularity from my perspective (see Figure 1; the photographs in the figure are from a previous task outside the scope of the present report). To investigate students' operations for comparing these extents of angularity, I asked the students which pair of wooden chopsticks were more open, Kacie's short chopsticks (left in Figure 1) or Camille's long chopsticks (right in Figure 1).

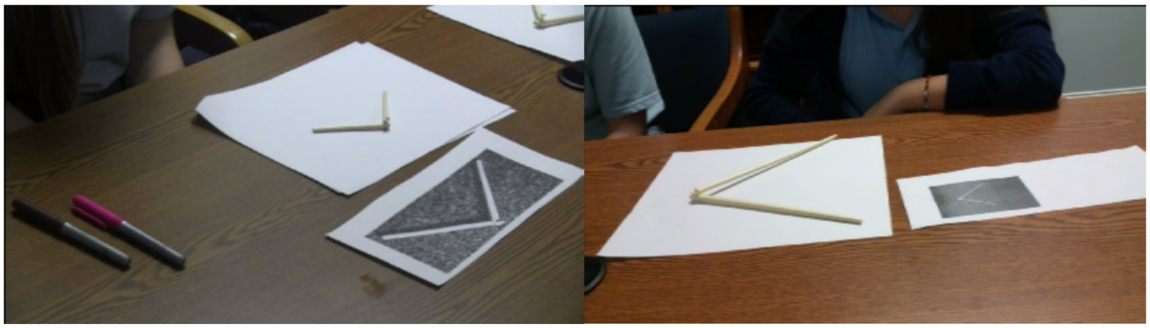

Figure 1: Kacie's (left) and Camille's (right) chopsticks as set from a previous task.

In this angular comparison task, Camille responded first and asserted Kacie's short chopsticks were more open. After a three second pause, Kacie hesitantly agreed the short chopsticks were more open. To understand how Camille was comparing extents of angularity, I asked her to explain her reasoning. Portions of Camille's response are described in Excerpt 1.

Excerpt 1: The truncation operation

C: I mean, I just kind of make this [long] one like if it was like that [short] one. So – kind of imagining it like that [*places marker across the top of her long chopsticks as if to render the long pair the same length as the short pair* (see Figure 2)]. So like this right here [*indicating the long chopsticks from vertex to marker*] is just like that [short] one, except that [short] one's more wider.

Figure 2: Camille truncates her chopsticks (right) to the length of Kacie's chopsticks (left)

From the interaction surrounding Excerpt 1, I inferred Camille mentally transformed the long chopsticks into the short chopsticks via the *truncation operation*, mentally shortening the long

chopsticks to a particular length, which was (in this case) the length as Kacie's short chopsticks. That Camille instantiated the operation is supported both by her placement of the marker (Figure 2 right) and her explanation, "I just kind of make this one like it was like that one." By truncating, Camille rendered the different lengths of the angle models inconsequential as she considered which angle model was more open.

Following further explanation from Camille, Kacie explained her reasoning and indicated a mental operation related to (but distinct from) truncation. Kacie's explanation and this second mental operation are described in the section below.

The Second Mental Operation: Hyper-truncation (Teaching Session #1)

As described above, Camille responded to the angular comparison task first, and Kacie paused before hesitantly agreeing with Camille. I infer Kacie's hesitation resulted from considering Camille's explanation, which likely differed from Kacie's own initial reasoning. Support for this inference appears in Excerpt 2, wherein Kacie explains her reasoning.

Excerpt 2: The hyper-truncation operation.

K: I was like looking at how open these were [*repeatedly tracing out the span of the long chopsticks*]. But then I realized that that probably wouldn't help me because it [the long pair] looks like it's more open than mine because it's bigger, like the chopsticks are longer than mine so it makes it look like it was more open. So then, I just kind of looked at like where they [*points to vertex of long pair*] cross or come together and saw that these [short pair] were like [*opens index fingers over the vertex*] not so like – I don't know how to explain it. [6 sec pause] That mine weren't so close together and these [long pair] were more – like there was littler space right here [*pointing to the interior near the vertex of the long pair as shown in Figure 3*] than there was in mine [*points similarly near the vertex of the shorter pair*].

Figure 3: Kacie points to the interior near the vertex of the long chopsticks

In Excerpt 2, Kacie indicated initially considering the span of the chopsticks and explained discarding this consideration due to differences in the lengths across the pairs of chopsticks. I suspect Kacie's attention to the differences in lengths was occasioned, at least in part, by Camille's assertion that the shorter chopsticks were more open than the longer chopsticks. Like Camille, Kacie indicated she had established a goal of rendering the lengths of the chopsticks inconsequential for comparing openness. Rather than truncate the longer chopsticks to match the length of the shorter chopsticks, Kacie mentally shortened both pairs of chopsticks as if trying to free them of length entirely. Although it is possible Kacie might have been imagining some previous experience involving angle markings (e.g., near-vertex arcs), this seems unlikely for two reasons: First, Kacie had not mentioned any such markings; Second, Kacie had not yet referred to the chopsticks as angles at this point in the teaching experiment. In discussing her comparison, Kacie was adamant that she limited her focus very near to the vertex: "I looked at like really close to where the chopsticks meet, but not like where they meet, like the space right after they meet." For this reason, I consider Kacie to have considered reducing the chopsticks to have sides of essentially infinitesimal length, which I refer to as *hyper-truncation*.

The Third Mental Operation: Elongation (Teaching Session #2)

In the pair's second teaching session and at my request, Camille re-demonstrated her use of a marker to shorten the long chopsticks to match the lengths of the short pair. Because both students had, to this point, indicated shortening side lengths but not extending them, I requested the pair consider if it was possible to think of a way to "turn the shorter pair into the longer pair."

Following this request, Camille suggested cutting material from another chopstick and adding it to the shorter pair. Providing them with markers and paper, I asked the students to draw how they would add to material to the short chopsticks. Camille positioned the shorter chopsticks within the longer so that the vertices and sides were adjacent. Using a marker, Kacie linearly extended one side of the short chopsticks to match the length of the corresponding side of the long chopsticks.

Camille's verbal description and Kacie's physical activity indicated each student had constructed an *elongation operation*, which I consider to be the cognitive inverse of the truncation operation. An individual instantiates elongation when they mentally extend an angle model's sides to an arbitrary but particular length. Prior to this point in the teaching experiment, neither student had spontaneously indicated an elongation operation in any tasks involving these physical angle models.

The Fourth Mental Operation: Hyper-elongation (Teaching Session #3)

Because Camille was absent, I worked on-on-one with Kacie during her third teaching session and aimed to further examine how Kacie was thinking about various attributes of the chopsticks. In particular, I sought to investigate whether Kacie differentiated angularity from spanned area (i.e., the area of the triangle formed by an angle model's sides and span). To accomplish this, I presented Kacie with a pair of chopsticks and sequentially placed bits of pipe cleaner near the chopsticks. From my perspective, some bits were contained in the spanned area and some that were not. In our conversation, we referred to the bits of pipe cleaner as "dots," angularity as "openness," and spanned area as "area." I asked Kacie to consider whether each dot was in the openness and area of the chopsticks, and also to explain her reasoning.[12]

12 Footnote: I am grateful to Les Steffe for suggesting the use of this task for examining Kacie's operations for extending the sides of the chopsticks. After the teaching experiment, I discovered Silfverbeg & Joutsenlathti (2014) investigated prospective teachers' conceptions of angles as objects using a similar task.

When I placed the first dot within the spanned area of the chopsticks (from my perspective), Kacie explained that the dot was in both the openness and the area of the chopsticks (see "1" in Figure 4). When I placed a second dot beyond the endpoints of the chopsticks (see "2" in Figure 4), Kacie explained, "I wouldn't consider it in the area, but I would consider it in the openness." When I asked Kacie for additional elaboration, she continued, "The way I think of it it's like these [sides] are still going on [*motions as if extending the sides of the chopsticks away from the vertex*]." At my request, Kacie used a marker to further illustrate her thinking and she drew in the segments shown in Figure 4. Although Kacie did not use, for example, arrows to visually indicate the sides of the angle model extended indefinitely, she clearly verbally indicated this was her intention. I refer to the mental indefinite extension of both sides of an angle model as *the hyper-elongation operation*. In this task, the hyper-elongation operation supported Kacie's differentiation of at least two attributes: spanned area and angularity.

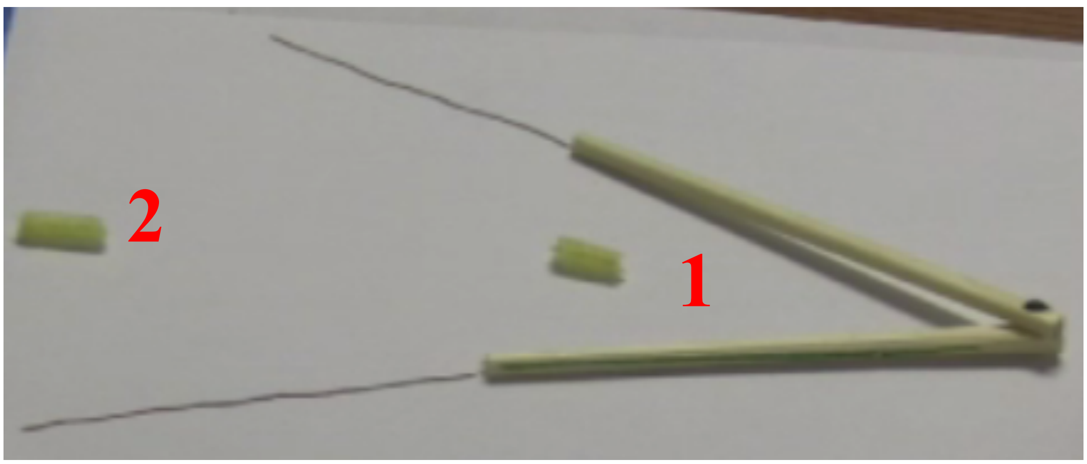

Figure 4: Kacie hyper-elongates the chopsticks

Kacie's activities indicated she had made at least a temporary distinction between two different attributes: the openness and the spanned area. Previously in the teaching experiment, Kacie had used truncation to compare congruent spanned areas; by truncating the long chopsticks, Kacie had rendered the lengths of the angle models' sides *inconsequential*.

Discussion and Concluding Remarks

In the preceding sections, I presented four mental operations the students enacted to alter lengths in angular contexts while preserving angularity: truncation, hyper-truncation, elongation, and hyper-elongation (Figure 5). In truncation and elongation, the sides of an angle model are mentally shortened or extended, respectively, to an arbitrary particular length (e.g., the length of another angle model). For hyper-truncation and hyper-elongation, the sides of angle model are mentally shortened or extended indefinitely; thus, hyper-truncation and hyper-elongation render an angle model's sides either infinitesimal segments or infinite rays, respectively.

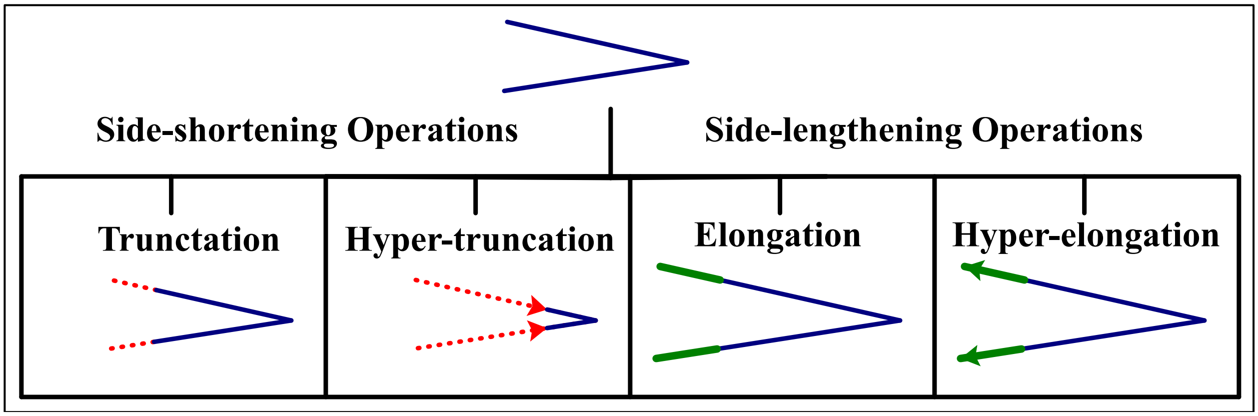

Figure 5: Four length-altering, angularity-preserving mental operations

Because comparing extents of angularity indicated by two perceptually available angle models necessarily involves considering finite, but not necessarily congruent lengths, I conjecture that these length-altering, angularity-preserving operations play a critical role in initial quantifications of angularity (i.e., those quantifications that involve gross or extensive quantitative operations; see author, year). As demonstrated in the findings above, these operations supported Kacie and Camille as they compared extents of angularity and worked to differentiate angularity from other attributes. Because I observed side-shortening operations to emerge spontaneously from students' goal-directed comparative activities in the teaching experiment, I conjecture side-shortening operations may spontaneously develop more naturally than side-lengthening operations, which required more targeted and intentional interventions for students to indicate during the teaching experiment. Furthermore, side-shortening operations require mentally discarding readily available perceptual material whereas the side-lengthening counterparts requires the mental insertion of material that is not perceptually available. In addition to systematically investigating both of these conjectures, future studies should examine the development of the four mental operations described in this report and others like them, particularly with students who have yet to receive classroom instruction on angle measure, as well as the affordances and limitations each operation occasion in students' quantifications of angularity.

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